Appendix V. Uncertainties and Error Propagation

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A. Introduction

In science, the terms *uncertainties* or *errors* do not refer to mistakes or blunders. Rather, they refer to those uncertainties that are inherent in all measurements and can never be completely eliminated. In some fields (*e.g. certain areas of astronomy and cosmology*), uncertainties may be measured in orders of magnitude while in other fields (*e.g. precision spectroscopy*) uncertainties may be less than parts per million, PPM. A large part of a scientist's effort is devoted to understanding these uncertainties (*error analysis*) so that appropriate conclusions can de drawn from variable observations.

A common complaint of students is that the error analysis is more tedious than the calculation of the numbers they're trying to measure. This is generally true. However, measurements can be quite meaningless without knowledge of their associated errors. If you are told that Sue is 162 cm tall and Beth is 165 cm tall you might conclude that Beth is taller than Sue. But if you then learn that the measurements had errors of ±5 cm, you should realize that you can't determine who is taller. A more precise measurement is required before you can make this comparison. In science and engineering, numbers without accompanying error estimates are suspect and possibly useless. **For every measurement, you must record the uncertainty in the measured quantity**.

Experimental errors may be divided into two classes: *systematic errors* and *random errors*. These are illustrated in Figure 1 for two sets of data points which are theoretically predicted to lie on the illustrated straight lines. The data in Figure 1a are relatively *precise*. They exhibit small *random errors* and therefore have small fluctuations about a straight line. However, there is a systematic shift of the data points away from the expected straight line. We attribute this effect to a *systematic error* in the measurements. The data in Figure 1b exhibit large random fluctuations, but they bracket the expected straight line. These data have large *random errors* but small systematic errors.

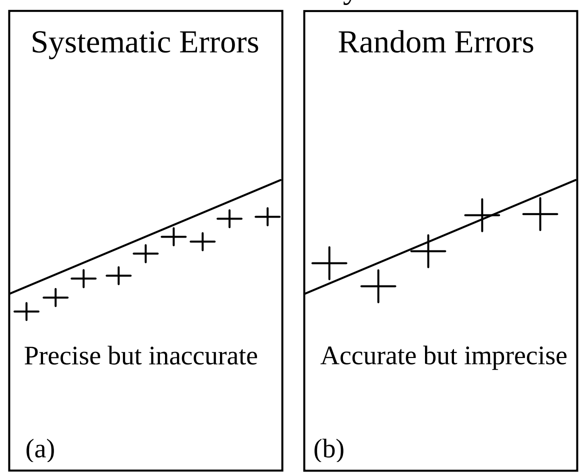

Figure 1: Examples of data where (a) systematic errors are larger than random errors and (b) random errors are larger than systematic errors

A.1. Systematic Errors

Systematic errors tend to produce inaccurate results by introducing a common shift into measured values. This shift can be an offset or a percentage change. For example, if your wooden meter stick had the first mm cut off, there would be an offset in all of your measurements. If, on the other hand, the humidity in the room had caused the meter stick to expand by 1%, there would be a percentage error in all of your measurements. Systematic errors may be caused by incorrect calibration of the measuring equipment and often can be reduced by readjusting or recalibrating equipment. Systematic errors might also be caused by not correctly accounting for some phenomena in your model and might be corrected by adopting a more sophisticated model. The effects of systematic errors on an experiment should be estimated and, if they are important, they should be reported separately from the random errors in the experimental results.

Note that the systematic errors have no effect on the slope of the graph in Figure 1a, but lead to an incorrect value for the intercept. Such systematic errors may or may not be important in an experiment, depending on whether the slope or the intercept (*or both*) provide critical information. (In other experiments, systematic errors could lead to an incorrect value for the slope.)

A.2 Random Errors

There are many sources of random errors, such as equipment limitations, reading uncertainties, and statistical fluctuations. Common examples are the uncertainties in reading scale divisions of an analog voltmeter or a ruler and statistical fluctuations in counting rates from random processes. We can often reduce these uncertainties by repeated measurements. However, while it may be possible to reduce random errors, they can never be completely eliminated.

B. Determination of Uncertainties

B.1. Uncertainties from Statistics

If we make repeated measurements of the same quantity, we can apply statistical analysis to study the uncertainties in our measurements. This type of analysis yields *internal errors,* i.e., the uncertainties are determined from the data themselves without requiring further estimates. The important variables in such analyses are the *mean*, the *standard deviation* and the *standard error* (also known as the *error in the mean*).

B.1.1 Mean or Average

To obtain the *best estimate* of a measured quantity from N measurements of the quantity, calculate the *mean* $\langle x \rangle$ or average of the measurements.

$$\langle x \rangle = \frac{\sum_{i=1}^{N} x_i}{N}$$

B.1.2. Standard Deviation

The standard deviation σ describes the scatter of measurements about the average, and is given by

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N} \left(x_i - \langle x \rangle \right)^2}{N-1}}$$

The *variance* is the square of the standard deviation; $v = \sigma^2$.

B.1.3. Standard Error

The *standard error* or *error in the mean is*

$$\delta = \frac{\sigma}{\sqrt{N}}\,.$$

This quantity is also referred to as the *standard deviation of the mean*, because it is an estimate of the *standard deviation* of the distribution of means that would be obtained if the mean were measured many times. Taking more measurements of a given quantity might not improve the *standard deviation*, but it should make the *standard error* smaller (*scaling it as* $N^{-½}$).

So how do you know whether the *standard deviation* or the *standard error* is the more important quantity? It depends on the question. If you want to know where a measurement is likely to fall compared to the mean value, the *standard deviation* tells you this. On the other hand, if you want to know how well you have determined the average value itself, you need to find the *standard error* (*standard deviation of the mean*). You will usually, but not always, be most interested in the latter.

As an example, let's say that everyone in the class is asked to take 20 measurements of the height of a certain lab TA. Each student can determine his or her own *average* value and *standard deviation*. The *standard error* will be an indication of the spread in the average values reported by all the students. It should be our best overall estimate of how well we know the TA's height.

One could also combine all the readings from all of the students into one large file and calculate its *mean* and *standard deviation*. These might be very similar to the values reported by individual students and have a similar spread in values. Only when you consider the new *standard error* would you realize that the measurement really has been improved by adding a lot more data.

B.1.4. Two Variables

We can readily extend the concept of the standard deviation to the measurement of two variables where our *N* measurements of *x* and *y* are to be compared to the function $y=f(x)$. The standard deviation for such measurements would be defined as

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N}\left(y_i - f\left(x_i\right)\right)^2}{N-m}}$$

where *m* is the number of free parameters determined from the data. For a linear relation, with the intercept and slope determined from the data by a least-squares fit, $m = 2$.

B.2. Error Estimates

When we have made only a few observations, the laws of probability are not applicable to the determination of uncertainties. The number of observations in a student laboratory may be too small to justify using the standard deviation to estimate the uncertainty in a measurement. However, it is usually possible from an inspection of the measuring instruments to set limits on the range in which the true value is most likely to lie.

Consider a ruler graduated in centimeters with fine rulings in millimeters as shown in Figure 2 (with only the 5 and 6 cm marks visible). We wish to determine the position of the arrow. We are certain the arrow is between 5.3 and 5.4 cm and we should be able to estimate its position to a fraction of a division. A reasonable estimate might be 5.34 ± 0.02. (*In reading most scales we should attempt to estimate some fraction of the smallest division, usually between one half to one tenth of the scale division.*)

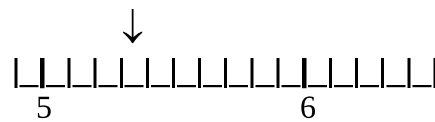

Figure 2: Example ruler reading

Uncertainties estimated in this way are referred to as *external errors*, i.e., estimating the uncertainties requires additional steps beyond making the measurements.

For a complete uncertainty analysis, both internal errors and external errors should be calculated and checks should be made that the results are consistent. In our experiments students will usually be instructed to choose one particular method or the other.

C. Error Propagation

In many cases, the quantity that we wish to determine is derived from several measured quantities. For example, suppose that we have measured the quantities $t \pm \delta_t$ and $y \pm \delta_y$ (*the δ's refer to the relatively small uncertainties in t and y*). We have determined that

$$y = 5.32 \pm 0.02 \text{ cm}$$
$$t = 0.103 \pm 0.001 \text{ s} \quad (1)$$

We can find *g* from the relation

$$g = g(y,t) = 2y/t^2, \quad (2)$$

which yields $g = 10.02922\ldots\text{m/s}^2$. To find the uncertainty in *g* caused by the uncertainties in *y* and *t*, we consider *separately* the contribution due to the uncertainty in *y* and the contribution due to the uncertainty in *t*.

Each contribution may be considered separately so long as the variables y and t are independent of each other. We denote the contribution due to the uncertainty in y by the symbol δ_{gy} (read as "delta-g-y" or "uncertainty in g due to y"). The total error in g is obtained by combining the individual contributions in quadrature:

$$\delta_g = \sqrt{\delta_{g_t}^2 + \delta_{g_y}^2} \qquad (3)$$

The basis of the quadrature addition is an assumption that the measured quantities have a Gaussian distribution about their mean values. (*Distribution functions are described in Appendix VI.*) When two (*or more*) independent Gaussians are added, the width of the new, combined distribution of values is given by this same quadrature rule. The δ*'s* describe the width of this distribution. This rule is the same as the rule for adding the lengths of vectors that are independent (*i.e. at right angles to each other*).

This quadrature addition may be used when the function depends on more measured quantities. For a function $f(a,b,...,z)$,

$$\delta_f = \sqrt{\delta_{fa}^2 + \delta_{fb}^2 + \cdots \delta_{fz}^2}\,. \qquad (4)$$

However, rather than blindly applying this formula, you may avoid needless computation by estimating separately the contribution to the uncertainty in the result from each of the individual variables and to ignore any terms that are much smaller than the largest terms. Because we add the *squares* of the individual contributions, relatively small terms have a very small effect on the total uncertainty.

There are two methods by which one may calculate δ_{gy} and δ_{gt}, the contributions of y and t to the uncertainty in g. In each case, the basic idea is to determine by how much g would change if y (or t) were changed by its uncertainty.

C.1. Derivative Method

The variation of a function f with respect to a variable x is equivalent to taking the first term in the Taylor series expansions of f with respect to x:

$$\delta_{fx} = \left(\frac{\partial f}{\partial x}\right)\delta_x \qquad (5)$$

The derivative in Eq. 5 ($\partial f/\partial x$) is a *partial* derivative. You may not have encountered partial derivatives yet in your math class. Simply put, when taking a partial derivative with respect to one variable, treat any other variables as constants.

Since each uncertain variable will *increase*, not decrease the final uncertainty, we will usually quote the uncertainty in f due to the uncertainty in x as the absolute value of Eq. 5., *i.e.*,

$$\delta_{fx} = \left|\frac{\partial f}{\partial x}\right|\delta_x \qquad (5a)$$

The individual contribution to the uncertainty in f from a measured uncertainty in x is the product of the uncertainty in x with the partial derivative of f with respect to x. The total error in f is obtained by combining the individual contributions in quadrature, as given in Eq. 4.

For the example given in Eq. 2, the corresponding formulae are

$$\delta_{gt} = |\partial g/\partial t|\delta_t = |-4y/t^3|\delta_t \qquad (6)$$
$$\delta_{gy} = |\partial g/\partial y|\delta_y = |2/t^2|\delta_y \qquad (7)$$

so that

$$\delta_g = \sqrt{\left(\frac{4y}{t^3}\delta_t\right)^2 + \left(\frac{2}{t^2}\delta_y\right)^2} \qquad (8)$$

Substituting in the values from Eq. 1 yields a final answer:

$$\delta_g = \sqrt{\left(\frac{4\times 5.32}{(0.103)^3}0.001\right)^2 + \left(\frac{2}{(0.103)^2}0.02\right)^2} = 19.8\ \text{cm/s}^2$$

C.2. Computational Method

Equations 4 and 5 represent the beginning of the formal method of error propagation. It is often a good estimate if we instead calculate the variations directly, thereby avoiding the need to take derivatives.

We can approximate Eq. 5a by a finite difference such as

$$\delta_{f,x} = \left| f\left(x+\delta_x,\ldots\right) - f\left(x,\ldots\right)\right| \quad (9)$$

Consider Eq. 2. We replace Eqs. 6 and 7 by

$$\delta_{gt} = |g(t+\delta_t, y) - g(t,y)| \qquad (10)$$

$$= |2y/(t + \delta_t)^2 - 2y/t^2|$$

and

$$\delta_{gy} = |g(t, y+\delta_y) - g(t,y)| \qquad (11)$$

$$= |2(y+\delta_y)/t^2 - 2y/t^2|.$$

We again apply Equation 2 to obtain the total uncertainty in g. (*For* $\delta_t << t$, *Equations 6 and 10 are equivalent, and for* $\delta_y << y$, *Equations 7 and 11 are equivalent*.)

Note that in both methods, it is essential that the variations be performed separately and that the results be added in quadrature.

C.3.Simple Error Propagation

Often you will simply add, subtract, multiply or divide measured values and it is helpful to know how to quickly calculate the associated errors.

Addition & Subtraction

If $g = g(y,t) = y + t$

then $\delta_{gy} = \delta_y$ and $\delta_{gt} = \delta_t$

so that $\delta_g = \sqrt{\delta_{g_t}^2 + \delta_{g_y}^2} = \sqrt{\delta_t^2 + \delta_y^2}$

Stated more simply, if you are adding two values, simply add their associated uncertainties in quadrature to obtain the uncertainty in the sum. The **same** principle holds if you are subtracting two numbers; **add** their uncertainties in quadrature to find the uncertainty in the difference. There are no *negative* uncertainties, uncertainties are always positive numbers and always add.

Note also that if you are adding a constant, such as 1, or a quantity with a very small uncertainty to some other quantity, the result above shows that the final uncertainty is simply the uncertainty in that other quantity.

Multiplication

If $g = g(y,t) = y\,t$

then $\delta_{gy} = t\delta_y$ and $\delta_{gt} = y\delta_t$

so that

$$\delta_g = \sqrt{\delta_{g_t}^2 + \delta_{g_y}^2} = \sqrt{y^2\delta_t^2 + t^2\delta_y^2}$$

Division

If $g = g(y,t) = y/t$

then $\delta_{gy} = \delta_y(1/t)$ and $\delta_{gt} = (y/t^2)\delta_t$

so that

$$\delta_g = \sqrt{\delta_{g_t}^2 + \delta_{g_y}^2} = \sqrt{\left(\frac{y}{t^2}\right)^2\delta_t^2 + \left(\frac{1}{t}\right)^2\delta_y^2}$$

It's also worth pointing out that *fractional* or *percentage* uncertainties in multiplication and division behave much like absolute uncertainties in addition. In other words, if $g = yt$,

$$\frac{\delta_g}{g} = \frac{\sqrt{y^2\delta_t^2 + t^2\delta_y^2}}{yt} = \sqrt{\left(\frac{\delta_t}{t}\right)^2 + \left(\frac{\delta_y}{y}\right)^2}$$

with the same result holding if $g = y/t$.

If either y or t is a constant or has a relatively small fractional uncertainty, then it can be ignored and the total uncertainty is just due to the remaining term.

Furthermore, if one of the measured quantities is raised to a power, the fractional uncertainty due to that quantity is merely multiplied by that power before adding the result in quadrature. For our original example of $g = 2y/t^2$,

$$\frac{\delta_g}{g} = \frac{\sqrt{\left(4y\delta_t/t^3\right)^2 + \left(2\delta_y/t^2\right)^2}}{2y/t^2} = \sqrt{\left(\frac{2\delta_t}{t}\right)^2 + \left(\frac{\delta_y}{y}\right)^2}$$

For the values in our example, $\delta_y/y = 0.5\%$ and $\delta_t/t = 1\%$ (so $2\delta_t/t = 2\%$), so we can see that the contribution from the uncertainty in y is negligible compared to the contribution from t. We can therefore conclude that the fractional uncertainty in our measured result for g is about 2%:

$$g = 10.0 \pm 0.2 \text{ m/s}^2$$

D. Significant Figures

Significant figures are those figures about which there exists no or very little uncertainty. In the example illustrated in Figure 2, the "5" and "3" are known exactly, while the "2" is known to some degree of certainty. Thus, the number has 3 significant figures. Care should be taken to distinguish between significant figures and decimal places. The scale reading could have been expressed as 0.0532 m, but it would still have 3 significant figures. The zeros preceding the "5" are "place markers" and are not significant figures. On the other hand, quoting the result as 5.320 cm would imply that the 2 is well known while the trailing 0 is also known, but with some degree of uncertainty. A trailing zero after the decimal point is thus considered to be significant.

Reporting Results (Measurement Intervals)

It is important to report your results with the correct number of significant figures. Suppose you have obtained from your calculations

$g = 9.98328 \text{ m/s}^2$ with $\delta_g = 0.067695 \text{ m/s}^2$.

Begin by rounding the uncertainty in your result to **one significant figure** *(or possibly 2)*, i.e., $\delta_g = 0.07 \text{ m/s}^2$. (*Since the uncertainty only tells you how well you have measured your result, it doesn't make sense to quote an uncertainty to more than one or two significant figures.*) Then quote g to the same number of decimal places, i.e.,

$g = (9.98 \pm 0.07) \text{ m/s}^2$,

not

$g = (9.98328 \pm 0.067695) \text{ m/s}^2$ *nor*

$g = (9.98328 \pm 0.07) \text{ m/s}^2$ *nor*

$g = (10 \pm 0.07) \text{ m/s}^2$.

The bold example above has a form sometimes referred to as a **measurement interval**. If you are using scientific notation, **always use the same power of 10** for both the quantity and its uncertainty. For example, quote

$h = (6.4 \pm 0.3) \times 10^{-34}$ J·s,

not

$h = (6.4 \times 10^{-34})$ J·s $\pm (3 \times 10^{-35})$ J·s.

You must include the units.